2 Continuous time models

Any variable whose value changes over time in an uncertain way is said to follow a stochastic process. Such processes can be classified as discrete time or continuous time. Stochastic processes can also be classified as having continuous state space or discrete state space. In this part of the course, we aim to model the dynamics of the price of a stock via a continuous state space, continuous time stochastic process. Naturally, stock prices take discrete values (e.g. multiples of a penny), nevertheless, the continuous state, continuous time interpretation can be extremely useful and many important results (such as the Black-Scholes pricing formula) can be derived from this setting.

We begin by reviewing Brownian motion and geometric Brownian motion before considering some further topics.

2.1 Brownian motion

Definition 2.1 (stochastic process)

A *stochastic process* is a collection of random quantities $\{X_t, t \in T\}$ with state space $\mathcal{S}$ and index set T. We will consider only continuous state space, continuous time processes, that is with $\mathcal{S} \subseteq \mathbf{R}$ and $T \subseteq \mathbf{R}^+$.

The first such process we will consider as a model for stock price is Brownian Motion. This long-studied process was first observed by botanist Robert Brown in 1827 (hence the name). It was proposed as a model for asset price movements in 1900 by Louis Bachelier whilst governing laws were stated by Albert Einstein. Norbert Wiener proved many results including non-differentiability of sample paths. Consequently, a 1-d Brownian motion is often referred to as a *Wiener process*.

Definition 2.2 (standard Brownian motion)

Formally, $\{W_t, t \geq 0\}$ is a *standard Brownian motion* (B.M.) if W_t depends continuously on t, $W_t \in (-\infty, \infty)$ and the following 3 assumptions hold

(1) $W_0 = 0$ with probability 1,
(2) The increment $W_{t_2} - W_{t_1}$ is independent of the increment $W_{t_1} - W_{t_0}$ for all times $t_2 > t_1 > t_0 \geq 0$
(3) For all times $0 \leq t_1 < t_2 < \infty$, $W_{t_2} - W_{t_1} \sim \mathrm{N}(0, t_2 - t_1)$.

Important properties

- The process is *Markov*: $\{W_t\}$ has the property that future states are independent of the past states given the present state.

- Note that using Definition 2.2(1) and 2.2(3) gives $W_t - W_0 = W_t \sim \mathrm{N}(0, t)$. Since we have used $W_0 = 0$, we may prefer to write $(W_t | W_0 = 0) \sim \mathrm{N}(0, t)$.

- For times $s < t$ define the *transition density* of the process by $p(w_t \,|\, W_s = w_s)$. Now note that
$$W_t = W_t - W_s + W_s \,.$$
From Definition 2.2(3), $W_t - W_s \sim \mathrm{N}(0, t - s)$ and so conditioning on $W_s = w_s$ gives
$$(W_t | W_s = w_s) \sim \mathrm{N}(w_s, t - s) \tag{1}$$
and hence the transition density is
$$f(w_t \,|\, W_s = w_s) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left\{ -\frac{(w_t - w_s)^2}{2(t-s)} \right\}, \; -\infty < w_t < \infty.$$
This is leads us to think about the process as a continuous time random walk – given a value w_s of the process at time s, the distribution of the process at a future time t is w_s plus some zero mean Gaussian noise.

- Standard Brownian motion can be generalised by scaling by a constant and shifting by a linear function of time. A *generalised Brownian motion* with *drift* a and *diffusion coefficient* b^2 is defined as
$$X_t = x_0 + at \,+\, b\,W_t \qquad a \in \mathbf{R},\, b^2 \in \mathbf{R}^+$$
For times $s < t$ we have that
$$\begin{aligned} X_t &= X_t - X_s + X_s \\ &= x_0 + at \,+\, b\,W_t - x_0 + as \,+\, b\,W_s + X_s \\ &= X_s + a(t-s) + b(W_t - W_s)\,. \end{aligned}$$
Hence
$$(X_t | X_s = x_s) \sim \mathrm{N}\left(x_s + a(t-s)\,,\, b^2(t-s)\right)\,. \tag{2}$$
Note that $a = 0$ and $b = 1$ returns the standard Brownian motion process.

Example 2.1

Suppose the cash position of a company (measured in thousands of pounds) follows a generalised B.M. with drift $a = 20$ per year and variance 900 per year (i.e. diffusion coefficient $b^2 = 900$). Initially the cash position is 50. Write down the distribution of the cash position after 6 months, 1 year, 10 years.

Solution

Denote the cash position at time t by X_t. Let $x_0 = 50$. Using equation (2), after 6 months (=0.5 years)

$$\begin{aligned}(X_{0.5}|X_0 = 50) &\sim N(50 + 20 \times 0.5\,,\, 900 \times 0.5) \\ &= N(60\,,\,450)\end{aligned}$$

Similarly,

$$\begin{aligned}(X_1|X_0 = 50) &\sim N(70\,,\,900) \\ (X_{10}|X_0 = 50) &\sim N(250\,,\,9000)\end{aligned}$$

Note that 1. cash position can become negative (we interpret this as the situation where the company is borrowing funds) and 2. our uncertainty increases as the square root of how far ahead we are looking.

Example 2.2

For times $r < s < t < u$ show that

$$\mathrm{E}\left[(W_t - W_r)(W_u - W_s)\right] = t - s\,.$$

Solution

As it stands, the increments are not independent so we cannot simply take the expectation of each term in the product. (To see this, consider the intervals (r,t) and (s,u) which overlap.) So, we re-write the expression in such a way as to give a sum of products of independent terms. By adding and subtracting W_s and W_t we have

$$E\left[\{(W_t - W_s) + (W_s - W_r)\} \times \{(W_u - W_t) + (W_t - W_s)\}\right]\,.$$

Now, multiplying out gives

$$E\left[(W_t - W_s)^2 + (W_t - W_s)(W_u - W_t) + (W_s - W_r)(W_u - W_t) + (W_s - W_r)(W_t - W_s)\right]\,.$$

The last three terms in the sum involve pairs of independent increments. Hence, upon taking the expectation inside the brackets we see that all terms are zero except

$$E\left[(W_t - W_s)^2\right] = Var(W_t - W_s) = (t - s)$$

since $E(W_t - W_s) = 0$ from Definition 2.2(3).

2.1.1 Simulating/visualising Brownian motion

The expected length of the path followed by W_t in any time interval is infinite. Consequently, simulation of a full realisation of W_t on say $[0,T]$ is impossible. It is possible however to construct a *skeleton* of a sample path of W_t by discretising time and then simulating W_t at each time point using equation(2).

Split $[0,T]$ into $n+1$ *equidistant* points $0 = t_0 < t_1 < \dots t_n = T$. Let $t_{i+1} - t_i = \Delta t = T/n$. Consider a generalised B.M. X_t with drift a, diffusion b^2 and $X_0 = x_0$. The distribution of X_{t_1} conditional on $X_0 = x_0$ is Normal with mean $x_0 + a\Delta t$ and variance $b^2\Delta t$. We simulate from this distribution to obtain a realisation of X_{t_1}, namely x_{t_1}. Now simulate $X_{t_2}|X_{t_1} = x_{t_1} \sim \mathrm{N}(x_{t_1} + a\Delta t, b^2\Delta t)$. In general, at time t_i, simulate $X_{t_i}|X_{t_{i-1}} = x_{t_{i-1}} \sim \mathrm{N}(x_{t_{i-1}} + a\Delta t, b^2\Delta t)$.

Algorithmically:-

1. Initialise $X_0 = x_0$. Put $i := 1$
2. Simulate $X_{t_i}|X_{t_{i-1}} = x_{t_{i-1}} \sim \mathrm{N}(x_{t_{i-1}} + a\Delta t, b^2\Delta t)$
3. If $t_i = T$, stop otherwise put $i := i+1$ and go to step 2.

The following R function takes as arguments T, Δt, x_0, a and b, and returns a skeleton path of a generalised B.M.

```
genbm=function(T=20,dt=0.01,x0=0,a=0,b=1)
{
 n=T/dt
 simvec=vector("numeric",len=n+1)
 simvec[1]=x0
 for(i in 2:(n+1))
  {
    simvec[i]=rnorm(1,simvec[i-1]+a*dt,b*sqrt(dt))
  }
 simvec
}
```

Plot the path with

```
plot(ts(genbm(),start=0,deltat=0.01))
```

Figure 3 shows a single simulated realisation of a standard B.M. viewed at decreasing sampling intervals. Note that as $\Delta t \to 0$, the true process is obtained. Figure 4 shows 4 simulated realisations with varying drift a and diffusion b^2. Clearly, increasing a shifts the trajectory up (proportional to time) and increasing b causes the trajectory to vary more about its mean.

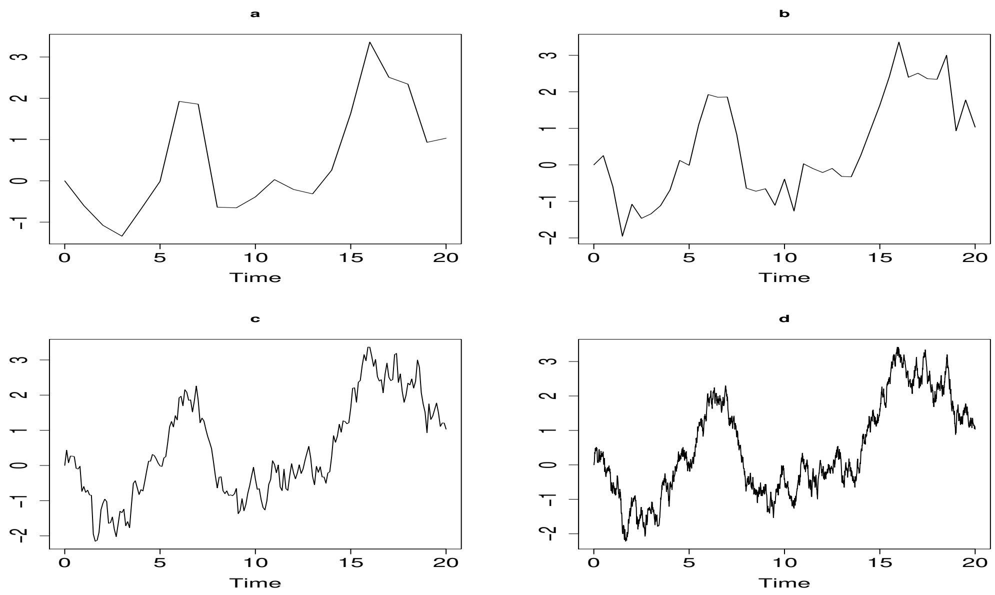

Figure 3: 1 simulated realisation of a standard B.M. with sampling frequency (a) $\Delta t = 1$, (b) $\Delta t = 0.5$, (c) $\Delta t = 0.1$ and (d) $\Delta t = 0.01$.

2.2 Lognormal distribution

In order to formulate a more realistic model of stock price we will first review the Lognormal distribution. Note that in this course, $\log = \log_e = \ln$.

Let Y be a Lognormal random variable with parameters m and v^2. Then, we write $Y \sim LN(m, v^2)$ with pdf

$$f_Y(y) = \frac{1}{y\sqrt{2\pi v^2}} \exp\left\{-\frac{(\log(y) - m)^2}{2v^2}\right\} \qquad y > 0\,.$$

The expectation of Y is

$$\mathrm{E}(Y) = e^{m+\frac{1}{2}v^2} \tag{3}$$

and the variance of Y is

$$\mathrm{Var}(Y) = e^{2m+v^2}\left(e^{v^2} - 1\right)\,. \tag{4}$$

Note that if $Y \sim LN(m, v^2)$, then $\log(Y) \sim N(m, v^2)$. Or, equivalently, if $X \sim N(m, v^2)$, then $Y = \exp(X) \sim LN(m, v^2)$. We can therefore obtain (3) and (4) by considering the moment generating function (mgf) of a $N(m, v^2)$ random variable, say X. Recall that this mgf (with arbitrary argument t) is

$$M_X(t) = \mathrm{E}\left(e^{tX}\right) = e^{mt+\frac{1}{2}v^2t^2}.$$

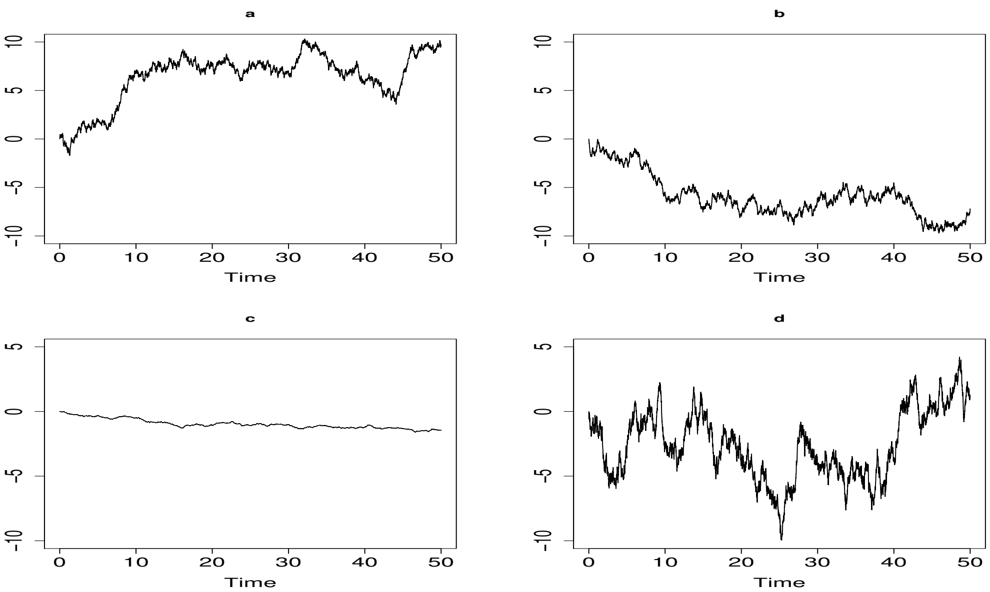

Figure 4: 4 simulated realisations of a generalised B.M. with (a) $a = 0.1$, $b = 1$ (b) $a = -0.1$, $b = 1$ (c) $a = 0$, $b = 0.1$ and (d) $a = 0$, $b = 2$.

Hence we obtain

$$\mathrm{E}\,(Y) = \mathrm{E}\left(e^X\right) = M_X(1) = e^{m+\frac{1}{2}v^2}.$$

The variance is obtained by first calculating $\mathrm{E}(Y^2) = M_X(2)$ and then using

$$\mathrm{Var}(Y) = \mathrm{E}(Y^2) - \{\mathrm{E}(Y)\}^2\,.$$

We can plot the density of Y for a range of m and v with the commands

```
y=seq(0,4,0.1)
plot(y,dlnorm(y,0,1),type="l",ylim=c(0,2))
lines(y,dlnorm(y,1,sqrt(2)),type="l")
lines(y,dlnorm(y,-1,1),type="l")
```

for which we obtain the Figure 5. Can you match up the distributions and their pdfs? Note that the pdfs are *right skewed*. Moreover, we have that

$$\mathrm{mode}(Y) = e^{m-v^2}\ <\ \mathrm{med}(Y) = e^m\ <\ \mathrm{E}(Y) = e^{m+\frac{1}{2}v^2}.$$

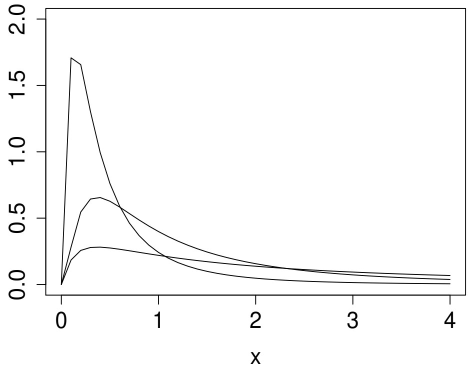

Figure 5: $LN(0,1), LN(1,2), LN(-1,1)$ pdfs

Finally note that the pdf $f_Y(y)$ of a Lognormal random variable $Y \sim LN(m, v^2)$ can be obtained from the pdf of a Normal random variable $X \sim N(m, v^2)$ as follows. Start with the cummulative distribution function of Y,

$$\begin{aligned} Pr(Y \leq y) &= Pr(\log(Y) \leq \log(y)) \\ &= Pr(X \leq \log(y)) \qquad \text{where } X \sim N(m, v^2) \\ &= F_x(\log(y)) \qquad \text{where } F_x(\cdot) \text{ denotes the cdf of } X. \end{aligned}$$

Differentiating with respect to y gives the pdf of Y as

$$\begin{aligned} f_Y(y) &= f_X(\log(y)) \times \frac{1}{y} \\ &= \frac{1}{y\sqrt{2\pi v^2}} \exp\left\{ -\frac{(\log(y) - m)^2}{2v^2} \right\}, \qquad y > 0 \end{aligned}$$

as required.

2.3 Geometric Brownian Motion

We have considered (generalised) Brownian motion as a model for cash position but not as a model for stock price. In fact, it would appear that (generalised) B.M. has two major flaws when used to model stock price:

1. When using B.M. the price of a stock would be a Normal random variable, and so it could be negative.

2. The assumption that the price difference over an interval of fixed length has the same Normal distribution no matter what the price at the beginning of the interval doesn't seem reasonable. For example, many people do not think that the probability a stock currently selling at 20 would drop to 15 or less (a loss of 25% or more) in one month should be the same as the the probability of a stock currently at 10 dropping to 5 or less in one month (a loss of 50% or more). Under generalised B.M. $Pr(X_t < 15|X_s = 20) = Pr(X_t < 5|X_s = 10) = Pr(X_t - X_s = -5)$.

The geometric Brownian motion model has neither of these flaws. Let us see why.

Definition 2.3

A continuous time stochastic process $\{S_t, t \geq 0\}$ is called a geometric Brownian motion (G.B.M.) (with parameters μ and σ^2) if each path $t \to S_t$ is a continuous positive function of t and

(1) $S_0 > 0$ is fixed,
(2) For all $0 \leq t_1 < t_2 < \infty$ the r.v. S_{t_2}/S_{t_1} is independent of $\{S_u, u \leq t_1\}$,
(3) For all $0 \leq t_1 < t_2 < \infty$ the r.v. $\log(S_{t_2}/S_{t_1})$ is normally distributed with mean $\left(\mu - \frac{1}{2}\sigma^2\right)(t_2 - t_1)$ and variance $\sigma^2(t_2 - t_1)$.

Important properties / comments

- When modelling stock price with G.B.M., the logarithm of the stock's price is a Normal random variable and so the model does not allow for negative stock prices.

- Since ratios of prices separated by a fixed length of time have the same distribution, G.B.M. makes the more reasonable assumption that it is the percentage change in price (and not the absolute change) whose probabilities do not depend on the present price.

- μ is known as the *mean rate of return* or *expected rate of return* and σ^2 is the *volatility.*

- Let $Y = S_t/S_u$ (for times $u < t$). Then Y follows the *Lognormal* distribution (and taking the log of S_t/S_u results in a Normal random variable with mean $(\mu - \sigma^2/2)(t-u)$ and variance $\sigma^2(t-u)$).
- Using equation (3) with $m = (\mu - \sigma^2/2)(t-u)$ and $v^2 = \sigma^2(t-u)$, the expectation of Y is
$$\mathrm{E}(Y) = e^{\mu(t-u)} . \tag{5}$$
- Using equation (4), the variance of Y is
$$\mathrm{Var}(Y) = e^{2\mu(t-u)} \left(e^{\sigma^2(t-u)} - 1 \right) . \tag{6}$$
- The quantity
$$\eta = \frac{1}{t} \log \left(\frac{S_t}{S_0} \right)$$
is known as the *continuously compounded rate of return* (or simply the *return*) realised between times 0 and t, and is so called since rearranging η gives
$$S_t = S_0 e^{\eta t} .$$
- G.B.M. (with parameters μ and σ^2) is related to the standard B.M. via the formula
$$S_t = S_u \exp \left\{ \left(\mu - \frac{1}{2}\sigma^2 \right) (t-u) + \sigma \left(W_t - W_u \right) \right\} \qquad u < t, \qquad S_0 > 0 \text{ fixed.} \tag{7}$$
Rewriting equation (7) with $u = 0$ gives
$$S_t = \exp(X_t), \qquad X_t = x_0 + \left(\mu - \frac{1}{2}\sigma^2 \right) t + \sigma W_t.$$
That is, X_t is a generalised Brownian motion with initial value $x_0 = \log(S_0)$, drift $a = \mu - \sigma^2/2$ and diffusion coefficient $b^2 = \sigma^2$.

We can show that equation (7) defines a G.B.M. by checking Definition 2.3. The continuity of sample paths of W_t gives continuity of sample paths of $\exp(X_t)$. Now note that

1. $S_0 = \exp(x_0)$ which is a fixed, positive value.
2. For all times $0 \le t_0 < t_1 < t_2 < \infty$,
$$\frac{S_{t_1}}{S_{t_0}} = \exp\{(\mu - 0.5\sigma^2)(t_1 - t_0) + \sigma(W_{t_1} - W_{t_0})\}$$
and
$$\frac{S_{t_2}}{S_{t_1}} = \exp\{(\mu - 0.5\sigma^2)(t_2 - t_1) + \sigma(W_{t_2} - W_{t_1})\}.$$
Now, independence of the Brownian increments on the RHS gives independence of the ratios on the LHS.

3. For all times $0 \leq t_0 < t_1 < \infty$,
$$\log\left(\frac{S_{t_1}}{S_{t_0}}\right) = (\mu - 0.5\sigma^2)(t_1 - t_0) + \sigma(W_{t_1} - W_{t_0})$$
which, using $W_{t_1} - W_{t_0} \sim N(0, t_1 - t_0)$ gives
$$\log\left(\frac{S_{t_1}}{S_{t_0}}\right) \sim N\left((\mu - 0.5\sigma^2)(t_1 - t_0)\,,\, \sigma^2(t_1 - t_0)\right)$$
as required.

Example 2.3

Suppose that the price of a particular stock follows a G.B.M. $\{S_t, t \geq 0\}$ with mean rate of return $\mu = 0.01$ per year and volatility $\sigma^2 = 0.04$ per year. If the initial price of stock is 100, find:

(a) $\mathrm{E}(S_{10})$;

(b) $\Pr(S_{10} > 100)$.

Solution

(a) We require
$$\begin{aligned} E(S_{10}) &= E\left(\frac{S_{10}}{S_0} \times S_0\right) \\ &= S_0 E(Y) \end{aligned}$$
where $Y = S_{10}/S_0$ follows a log-Normal distribution and so we can apply equation (5). We obtain
$$E(S_{10}) = 100e^{\mu(10-0)} = 100e^{0.1}\,.$$
Note that in general,
$$E(S_t) = S_0 e^{\mu t}$$
so the expected price grows like a fixed-income security with continuously compounded interest rate μ. This is why we call μ the rate of return.

(b) We have
$$\begin{aligned} Pr(S_{10} > 100) &= Pr\left(\frac{S_{10}}{S_0} > \frac{100}{S_0}\right) \\ &= Pr\left(\log\left(\frac{S_{10}}{S_0}\right) > \log(1)\right) \\ &= Pr(X > 0) \qquad \text{where } X \sim N\left((\mu - 0.5\sigma^2)10\,,\, \sigma^2 10\right) \equiv N(-0.1, 0.4) \end{aligned}$$

Hence we obtain

$$\begin{aligned} Pr(S_{10} > 100) &= Pr\left(Z > \frac{0.1}{\sqrt{0.4}}\right) \qquad \textit{where } Z \sim N(0,1) \\ &= 0.437 \end{aligned}$$

Example 2.4

Consider a stock with an initial price of 40, an expected return of 16% per annum, and a volatility of 4% per annum. Calculate a 95% confidence interval for the stock price in 6 months time, $S_{0.5}$.

Solution

Identify $\mu = 0.16$ *and* $\sigma = 0.2$. *Now, we know that*

$$\begin{aligned} \log\left(\frac{S_{0.5}}{S_0}\right) &\sim N\left((0.16 - 0.2^2/2) \times 0.5, 0.2^2 \times 0.5\right) \\ \Rightarrow \log\left(\frac{S_{0.5}}{S_0}\right) &\sim N(0.07, 0.02) \end{aligned}$$

Hence, with 95% confidence,

$$\begin{aligned} & 0.07 - 1.96 \times \sqrt{0.02} < \log\left(\frac{S_{0.5}}{S_0}\right) < 0.07 + 1.96 \times \sqrt{0.02} \\ \Rightarrow\ & \log(40) - 0.207 < \log(S_{0.5}) < \log(40) + 0.347 \\ \Rightarrow\ & 32.52 < S_{0.5} < 56.59 \end{aligned}$$

2.3.1 Simulating/Visualising Geometric Brownian Motion

Just as with Brownian motion, we can simulate a skeleton of a sample path of geometric Brownian motion by discretising time and using equation (7). As before, split $[0, T]$ into $n+1$ *equidistant* points $0 = t_0 < t_1 < \dots t_n = T$. Let $t_{i+1} - t_i = \Delta t = T/n$. Perform the following sequence of steps:-

1. Initialise $S_0 = s_0$. Put $i := 1$
2. Simulate $W_{t_i} - W_{t_{i-1}} \sim N(0, \Delta t)$
3. Put $S_{t_i} = s_{t_{i-1}} \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)\Delta t + \sigma\left(W_{t_i} - W_{t_{i-1}}\right)\right\}$

4. If $t_i = T$, stop otherwise put $i := i + 1$ and go to step 2.

The following R function takes as arguments T, Δt, s_0, μ and σ, and returns a skeleton path of a generalised B.M.

```
gbm=function(T=20,dt=0.01,s0=40,mu=0.1,sig=0.2)
{
 n=T/dt
 simvec=vector("numeric",len=n+1)
 simvec[1]=s0
 for(i in 2:(n+1))
  {
   simvec[i]=simvec[i-1]*exp((mu-0.5*sig*sig)*dt+sig*rnorm(1,0,sqrt(dt)))
  }
 simvec
}
```

Plot the path with

```
plot(ts(gbm(),start=0,deltat=0.01))
```

Figure 6 shows two simulated realisations of a geometric B.M.

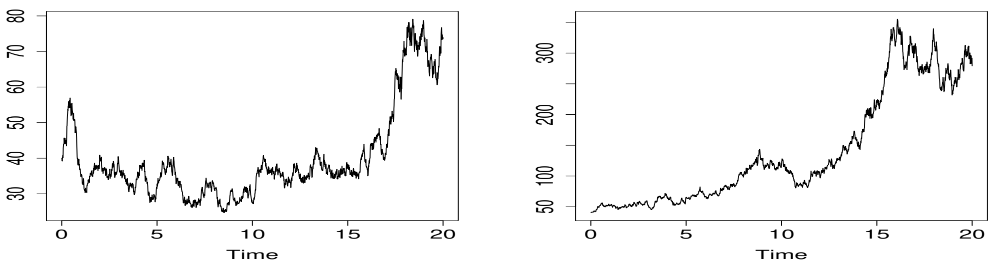

Figure 6: 2 simulated realisations of a geometric B.M. with $s_0 = 40$, $\mu = 0.1$ and $\sigma = 0.2$.

2.3.2 G.B.M. as a limit of simpler models (not examinable)

Partition the interval $[0, T]$ into n equal subintervals of size $\Delta t = T/n$ and consider a Binomial model for the price of a stock. That is, every Δt time units, the price either goes up by a factor u with probability p or goes down by a factor of d with probability $1 - p$. Fix μ and σ and set

$$u = e^{\sigma\sqrt{\Delta t}}, \qquad d = e^{-\sigma\sqrt{\Delta t}},$$

$$p = \frac{1}{2}\left(1 + \left(\frac{\mu}{\sigma} - \frac{\sigma}{2}\right)\sqrt{\Delta t}\right).$$

Now define a random variable Y_i taking the value 1 if the price goes up at time $i\Delta t$ and 0 if the price goes down. Plainly, the number of times the price goes up (in the first n time increments) is $\sum_{i=1}^n Y_i$ and the number of times it goes down is $n - \sum_{i=1}^n Y_i$. Hence, the stock price at time T can be expressed as

$$\begin{aligned} S_T &= S_0 u^{\sum_{i=1}^n Y_i} d^{n-\sum_{i=1}^n Y_i} \\ &= S_0 d^n \left(\frac{u}{d}\right)^{\sum_{i=1}^n Y_i}. \end{aligned}$$

Dividing by S_0 and taking logarithms gives

$$\begin{aligned} \log\left(\frac{S_T}{S_0}\right) &= n\log(d) + \log\left(\frac{u}{d}\right)\sum_{i=1}^n Y_i \\ &= \frac{-T\sigma}{\sqrt{\Delta t}} + 2\sigma\sqrt{\Delta t}\sum_{i=1}^{T/\Delta t} Y_i. \end{aligned} \qquad (8)$$

after using $n = T/\Delta t$ and the definitions of u and d. Now, taking smaller and smaller intervals, $\Delta t \to 0$, is equivalent to taking $n \to \infty$ and hence by the *central limit theorem*, $\sum_{i=1}^n Y_i$ becomes increasingly Normal. This implies that $\ln(S_T/S_0)$ in equation (8) becomes a Normal random variable. Taking expectations

$$\begin{aligned} \mathrm{E}\left[\log\left(\frac{S_T}{S_0}\right)\right] &= \frac{-T\sigma}{\sqrt{\Delta t}} + 2\sigma\sqrt{\Delta t}\sum_{i=1}^{T/\Delta t}\mathrm{E}(Y_i) \\ &= \frac{-T\sigma}{\sqrt{\Delta t}} + 2\sigma\sqrt{\Delta t}\frac{T}{\Delta t}p \\ &= \frac{-T\sigma}{\sqrt{\Delta t}} + \frac{T\sigma}{\sqrt{\Delta t}}\left(1 + \left(\frac{\mu}{\sigma} - \frac{\sigma}{2}\right)\sqrt{\Delta t}\right) \\ &= \left(\mu - \frac{\sigma^2}{2}\right)T. \end{aligned}$$

For the variance, we obtain

$$\begin{aligned} \mathrm{Var}\left[\log\left(\frac{S_T}{S_0}\right)\right] &= 4\sigma^2\Delta t\sum_{i=1}^{T/\Delta t}\mathrm{Var}(Y_i) \\ &= 4\sigma^2 Tp(1-p) \\ &\approx \sigma^2 T \qquad \text{since } p \approx 1/2 \text{ for small } \Delta t. \end{aligned}$$

Hence, we have shown that taking a simple Binomial model (of stock price) with smaller and smaller time periods results in the geometric Brownian motion. We can also verify this empirically. Consider the following R function that takes as arguments S_0, μ, σ, T and Δt, and returns a simulated value of $\log(S_T/S_0)$, by simulating from the Binomial model.

```
bin=function(T=2,dt=0.1,s0=40,mu=0.1,sig=0.2)
{
 n=T/dt
 sdt=sqrt(dt)
 u=exp(sig*sdt)
 d=exp(-sig*sdt)
 p=0.5*(1+(mu/sig-sig/2)*sdt)
 s=s0
 k=rbinom(1,n,p)
 s=s*u^(k)*d^(n-k)
 log(s/s0)
}
```

Consider an example with $T = 2$, $\mu = 0.1$ and $\sigma = 0.2$. For a Binomial model with 'small' time intervals, we should expect the distribution of $\ln(S_2/S_0)$ to be (approximately) Normal with mean $(\mu - 0.5\sigma^2)T = 0.16$ and variance $\sigma^2 T = 0.08$. The following function generates a predetermined number of simulated values of $\log(S_2/S_0)$,

```
bin2=function(T=2,dt=0.1,s0=40,mu=0.1,sig=0.2,sim=1000)
{
 simvec=vector("numeric",len=sim)
 for(i in 1:sim){
  simvec[i]=bin(T,dt,s0,mu,sig)
 }
 simvec
}
```

and we can then plot a histogram of these simulated values with

```
hist(bin2(),freq=F)
```

Figure 7 provides 4 such histograms generated with decreasing Δt.

2.3.3 Black-Scholes Pricing

In the final part of this Section, we derive the well known Black-Scholes formula, which gives (under the assumption that the price of a security evolves according to a G.B.M.) the unique no-arbitrage cost of a call option. The theory was developed in the early 1970s and its importance recognised in 1997, with the award of a Nobel prize for economics.

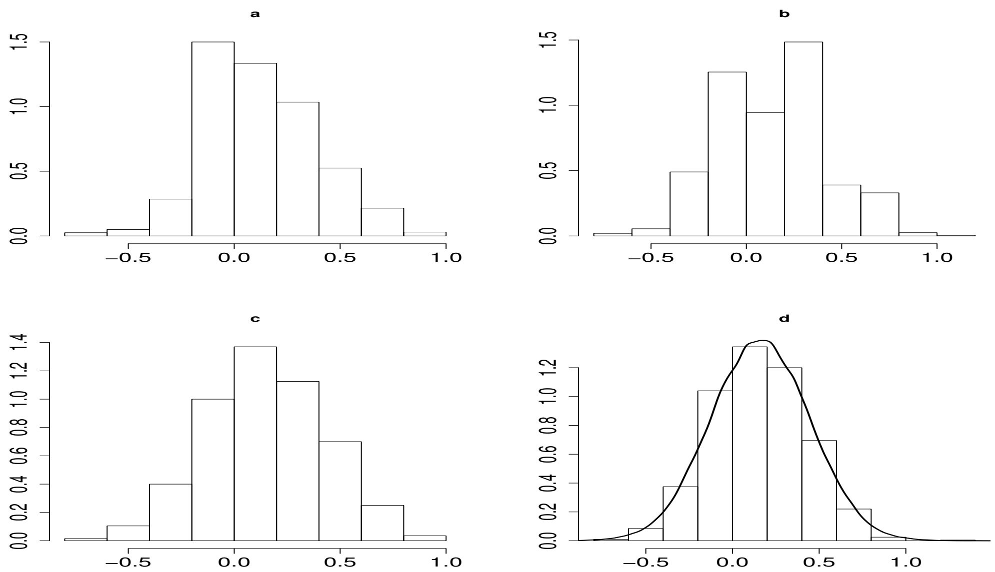

Figure 7: Histograms of 1000 simulations of $\log(S_2/S_0)$ from the binomial model with $\mu = 0.1$, $\sigma = 0.2$ and time intervals of (a) $\Delta t = 0.2$, (b) $\Delta t = 0.1$, (c) $\Delta t = 0.01$ and (d) $\Delta t = 0.005$. Plot (d) includes an overlay of the $\mathrm{N}(0.16, 0.08)$ density.

Consider an ECC with payoff $g(S_T)$ at time T. The no arbitrage fair price of this ECC is

$$e^{-rT}\mathrm{E}\left\{g(S_T)\right\}$$

where $\mathrm{E}(\cdot)$ should be an appropriate risk-neutral expectation. That is

"the discounted expected payoff at time T".

The motivation for this form of fair price is probably best understood in the context of gambling. It is helpful to imagine the payoff of the ECC as your total fortune at time T after gambling in a "fair" game. One might then expect the fair price for entering the game to be the expected payoff at time T. To take into account the money market, we multiply by the discount factor, e^{-rT}.

The simplest ECC has payoff S_T at time T. The fair price is therefore

$$e^{-rT}\mathrm{E}(S_T) = S_0 \tag{9}$$

since for no-arbitrage, the fair price of the ECC must coincide with its value at time 0. Under assumption of GBM,

$$\mathrm{E}(S_T) = S_0 e^{\mu T}$$

and we therefore must take $\mu = r$ for (9) to be satisfied. A G.B.M. with $\mu = r$ is known as *risk neutral* G.B.M. Under the risk-neutral G.B.M., $\log(S_T/S_0)$ is Normal with mean $(r - \sigma^2/2)T$ and variance $\sigma^2 T$.

Hence, the unique no-arbitrage cost, C_0, of a European call option with maturity T and strike price K is the discounted expected payoff at time T,

$$\begin{aligned} C_0 &= e^{-rT}\mathrm{E}\left[(S_T - K)^+\right] \\ &= e^{-rT}\mathrm{E}\left[(S_0 e^W - K)^+\right] \end{aligned}$$

where W is a Normal random variable with mean $(r - \sigma^2/2)T$ and variance $\sigma^2 T$. This equation can be explicitly evaluated to give the *Black-Scholes option pricing formula.*

Result 2.1 (Black-Scholes formula)

Under the assumption of R-N G.B.M., the fair price of a European call option with maturity T and strike price K is

$$C_0 = S_0\Phi(\omega) - Ke^{-rT}\Phi(\omega - \sigma\sqrt{T}), \qquad \text{where } \omega = \frac{rT + \sigma^2 T/2 - \log(K/S_0)}{\sigma\sqrt{T}} \tag{10}$$

and $\Phi(\cdot)$ is the standard Normal distribution function. *Recall that* log *is base* e.

Derivation of the Black-Scholes price (not examinable)

Let I be an indicator variable taking the value 1 if $S_T > K$ and 0 otherwise. By definition of the fair price C_0 of the European call option,

$$\begin{aligned} C_0 &= e^{-rT}\mathrm{E}\left([S_T - K]^+\right) \\ &= e^{-rT}\mathrm{E}\left(\max[0, S_T - K]\right) \\ &= e^{-rT}\mathrm{E}\left(I \times [S_T - K]\right) \end{aligned}$$

where I is the indicator variable defined above. Hence we obtain

$$C_0 = e^{-rT}\mathrm{E}\left[I \times S_T\right] - Ke^{-rT}\mathrm{E}\left[I\right] .$$

We now calculate the expectation of the indicator variable,

$$\begin{aligned} \mathrm{E}\left[I\right] &= 1 \times \Pr(S_T > K) + 0 \times \Pr(S_T \le K) \\ &= \Pr(S_T > K) \\ &= \Pr\left(\log\left[\frac{S_T}{S_0}\right] > \log\left[\frac{K}{S_0}\right]\right) \end{aligned}$$

where $\log\left[\frac{S_T}{S_0}\right] \sim N([r - \sigma^2/2]T, \sigma^2 T)$

$$= \Pr\left(Z > \frac{\log(K/S_0) - [r - \sigma^2/2]T}{\sigma\sqrt{T}}\right)$$

where $Z \sim N(0,1)$

$$= \Pr\left(Z < \frac{[r-\sigma^2/2]T - \log(K/S_0)}{\sigma\sqrt{T}}\right).$$

Now note that

$$\begin{aligned}\omega - \sigma\sqrt{T} &= \frac{rT + \sigma^2 T/2 - \log(K/S_0)}{\sigma\sqrt{T}} - \frac{\sigma^2 T}{\sigma\sqrt{T}} \\ &= \frac{[r-\sigma^2/2]T - \log(K/S_0)}{\sigma\sqrt{T}}.\end{aligned}$$

Hence,

$$\mathrm{E}[I] = \Phi(\omega - \sigma\sqrt{T})$$

where $\Phi(\cdot)$ denotes the CDF of a standard Normal random variable.

Now we just need $\mathrm{E}[I \times S_T]$. We start by writing S_T as

$$S_T = S_0 \exp\left\{(r-\sigma^2/2)T + \sigma\sqrt{T}Z\right\}$$

where $Z \sim N(0,1)$. We can then write our indicator variable as

$$I = \begin{cases} 1, & S_T > K \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & Z > \sigma\sqrt{T} - \omega \\ 0, & \text{otherwise} \end{cases}$$

Hence

$$\begin{aligned}\mathrm{E}[I \times S_T] &= \int_{\sigma\sqrt{T}-\omega}^{\infty} S_0 \exp\left\{(r-\sigma^2/2)T + \sigma\sqrt{T}z\right\} f_Z(z)dz \\ &= \int_{\sigma\sqrt{T}-\omega}^{\infty} S_0 \exp\left\{(r-\sigma^2/2)T + \sigma\sqrt{T}z\right\} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \\ &= \frac{1}{\sqrt{2\pi}} S_0 \exp\left\{(r-\sigma^2/2)T\right\} \int_{\sigma\sqrt{T}-\omega}^{\infty} \exp\left\{-(z^2 - 2\sigma\sqrt{T}z)/2\right\} dz \\ &= \frac{1}{\sqrt{2\pi}} S_0 e^{rT} \int_{\sigma\sqrt{T}-\omega}^{\infty} \exp\left\{-(z-\sigma\sqrt{T})^2/2\right\} dz \quad \text{(completing the square)} \\ &= S_0 e^{rT} \frac{1}{\sqrt{2\pi}} \int_{-\omega}^{\infty} e^{-y^2/2} dy \quad \text{by letting } y = z - \sigma\sqrt{T} \\ &= S_0 e^{rT} Pr(Z > -\omega) \\ &= S_0 e^{rT} \Phi(\omega) \quad \text{(by symmetry of the Normal pdf).}\end{aligned}$$

Hence we obtain

$$\begin{aligned}C_0 &= e^{-rT}\mathrm{E}\left[I \times S_T\right] - Ke^{-rT}\mathrm{E}\left[I\right] \\ &= e^{-rT}S_0e^{rT}\Phi(\omega) - Ke^{-rT}\Phi(\omega - \sigma\sqrt{T}) \\ &= S_0\Phi(\omega) - Ke^{-rT}\Phi(\omega - \sigma\sqrt{T})\end{aligned}$$

as required.

Comments

- Let C_0 and P_0 be the respective no-arbitrage costs of a European call and put option each with strike price K and maturity T. It follows from the put-call option parity formula (see Section 1) that P_0 is given by

$$P_0 = C_0 + Ke^{-rT} - S_0\,. \tag{11}$$

- Note that the no-arbitrage cost of the option depends on the underlying Brownian motion only through its volatility σ^2 (since r is known). In other words, to find the fair price of an option, we need only estimate σ^2.

Example 2.5

Consider an option with strike price $K = 29$ (in pounds) and maturity $T = 4$ months. Suppose that the current price of stock is $S_0 = 30$, the risk free interest rate is 5% and the volatility is 6.25% per annum.

(a) What is the price of the option if it is a European call?

(b) What is the price of the option if it is a European put?

Solution

(a) *Identify $T = 1/3$, $r = 0.05$, $\sigma^2 = 0.0625$, $\sigma = 0.25$. Let C_0 denote the price of the European call. Define P_0 similarly. Using the Black-Scholes formula (10) we have*

$$\begin{aligned}\omega &= 0.4225 \\ \Rightarrow C_0 &= 30\Phi(0.4225) - 29e^{-0.05\times 1/3}\Phi(0.4225 - 0.25\sqrt{1/3}) \\ &= 2.53\,.\end{aligned}$$

Hence the no-arbitrage price of the European call is £2.53.

(b) Using the put-call parity formula given by (11),

$$\begin{aligned} P_0 &= 2.53 + 29e^{-0.05\times 1/3} - 30 \\ &= 1.05\,. \end{aligned}$$

*Hence the no-arbitrage price the European put is £*1.05.

Properties of the Black-Scholes price

We have the following properties of the Black-Scholes price C_0:

1. C_0 *is an increasing function of* S_0. This means that if the other four variables (T, K, σ, r) remain the same, then the no-arbitrage cost of the option is an increasing function of the security's initial price. Showing this to be the case will be left as an exercise.

2. C_0 *is a decreasing function of* K. Showing this to be the case will be left as an exercise.

3. C_0 *is increasing in* T. A mathematical argument can be given but is beyond the scope of the course.

4. C_0 *is increasing in* σ. This at first might seem intuitive since the option holder will benefit from large prices at maturity time, while any additional price decrease below the strike price will not cause any additional loss. However, we must note that increasing σ also results in a decrease in the mean of an asset's price (under GBM). Nevertheless the result is true but a mathematical proof is beyond the scope of the course.

5. C_0 *is increasing in* r. Showing this to be the case will be left as an exercise.