Administrative Arrangements

- Course delivery using in-person lectures and problems classes.
- Two lectures per week, one problem class every other week (ish).
- Assessment is by exam (80%, January 2024) and assignments (20%), with four assignments (5% each) in total.
- The course page can be found on canvas at *https://canvas.ncl.ac.uk*

Books

- J. Hull: Options, Futures and Other Derivatives (Prentice-Hall, 2003)
- S. Ross: An Elementary Introduction to Mathematical Finance (CUP, 2003)
- M. Capinski, T. Zastawniak: Mathematics for Finance (Springer, 2003)
- S. Shreve: Stochastic Calculus for Finance 1 (Springer, 2004)
- J. Franke, W. Hardle, C. Hafner: Statistics of Financial Markets (Springer, 2004)

Important dates

Timetable week	Teaching week	w/c	Notes
4	1	25/09/23	
5	2	02/10/23	
6	3	09/10/23	
7	4	16/10/23	Ass. 1 due by 4pm on Oct. 20th
8	5	23/10/23	
9	6	30/10/23	Ass. 2 due by 4pm on Nov. 3rd
10	–	06/11/23	Enrichment week (no teaching)
11	7	13/11/23	
12	8	20/11/23	Ass. 3 due by 4pm on Nov. 24th
13	9	27/11/23	
14	10	04/12/23	Ass. 4 due by 4pm on Dec. 8th
15	11	11/12/23	
–	–	25/12/23	Xmas!

Table 1: Schedule

Late work policy

For normal written coursework, a deadline extension of up to 7 days can be requested (by means of submitting a PEC form); work submitted within 7 days of the deadline without good reason will be marked for reduced credit (following the University sliding scale). You should note that no work can be accepted more than 7 days after the original deadline; where work cannot be submitted by this time, the PEC Committee may agree instead to 'discount' or 'exempt' the work (although this would not be routine). For any time-limited assessments (e.g. tests open for 24 hours or less, including NUMBAS tests), rescheduling can be requested (by means of submitting a PEC form). Late work cannot be accepted for NUMBAS assessment; however, a deadline extension can be requested (by means of submitting a PEC form).

For details of the policy (including procedures in the event of illness etc.) please consult the Mathematics, Statistics & Physics Community pages on Canvas, under: Assessment Information, Late Work and Missed Assessments.

Course outline in brief

The course comprises five topics:

1. Risk-free and risky assets
    - Interest, compounding of interest
    - Options of European and American type
2. Continuous time models of stock price / cash position
    - Log-Normal distribution
    - Brownian motion, Geometric Brownian motion
    - Black-Scholes pricing
3. Estimating Volatility
    - Using historic data
    - Implied volatility
4. Exotic options and Monte Carlo simulation
    - Lookback, barrier and Asian options
    - Pricing using simulations from the model
5. Introduction to Itô calculus
    - Itô integral
    - Stochastic differential equations (SDEs)
    - Models of interest rate

Disclaimer

These notes may cover material that is not covered in the lectures. They may also omit some material that is covered. If you find any typos please let me know!

Revision

The majority of the course will use probability results for continuous random variables.

Probability Density Functions

Let X be a random variable (r.v.). We say that X is *continuous* if the probability of any fixed value x is 0, i.e. $Pr(X = x) = 0$, while the probability that X takes one of the values in some interval $[a, b]$, or (a, b), may be positive. We describe the probability law of such a variable in terms of its *distribution function* (d.f.) $F(x) = Pr(X \leq x), \ -\infty < x < \infty$. As we know, F is always right-continuous and monotone, increasing (in fact, non-decreasing) from value 0 at $-\infty$ to value 1 at ∞.

We assume more, that the d.f. F obeys a *density function*, say f. This means that

$$F(x) = \int_{-\infty}^{x} f(u)du; \ F'(x) = f(x) \ \text{ for all } x \in (-\infty, \infty).$$

Recall that f is a non-negative function and the total integral of f is equal to 1.

With F or f at hand, the probability that X takes a value in a given interval $[a, b], \ a < b$, is

$$Pr(a < X \leq b) = F(b) - F(a) = \int_{a}^{b} f(x)dx.$$

The *mean value* (expectation) of a 'nice' function $g(X)$ of the r.v. X is defined by

$$E[g(X)] = \int g(x)f(x)dx \ \text{ assuming it's finite }.$$

Transformations of Random Variables

Consider a random variable X with p.d.f. $f_X(x)$. The p.d.f. of an arbitrary differentiable invertible transformation $Y = g(X)$ can be deduced as

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| .$$

Note that the term $\left| \frac{d}{dy} g^{-1}(y) \right|$ is known as the "Jacobian" of the transformation.

Normal Random Variables

- We say that the r.v. X has a normal distribution with parameters μ and σ^2, $X \sim \mathrm{N}(\mu, \sigma^2)$, if X has the density function

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right], \ -\infty < x < \infty; \ \mu \text{ is any real, } \sigma > 0.$$

  This density is also called normal or Gaussian. The expectation and the variance of X are

$$E(X) = \mu, \quad \mathrm{Var}(X) = \sigma^2.$$

  The graph of $f(x)$, $-\infty < x < \infty$, is the familiar bell-shaped curve, symmetric about the axis $x = \mu$.

- A normal r.v. is called *standard* if $E(X) = 0$ and $\mathrm{Var}(X) = 1$. In this case we use the notation $Z \sim \mathrm{N}(0, 1)$. That is to say, the density of Z is

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \ -\infty < x < \infty.$$

  The graph of ϕ is symmetric about the y-coordinate axis. The corresponding d.f. is

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy, \ -\infty < x < \infty$$

  It is is called the standard normal (or Gaussian) distribution function.

Useful Properties of Normal Random Variables

- *Linear Combinations:* If the r.v. X is normal, then so is $aX + b$, where a, b are constants. If X has mean μ and variance σ^2, then $Z = (X - \mu)/\sigma$ is standard normal. (Can you check this?) This fact enables us to express probabilities related to X in terms of Φ.

- *Independence of random variables:* We say that the r.v.s $X_1, \ldots, X_n$ are *independent* if for arbitrary intervals $I_1, \ldots, I_n$, where $I_j = [a_j, b_j]$, closed or open, we have

$$P[X_1 \in I_1, \ldots, X_n \in I_n] = P[X_1 \in I_1] \cdot \ldots \cdot P[X_n \in I_n].$$

  If independent r.v.s X_j are normal with parameters μ_j, σ_j^2, $j = 1, \ldots, n$, then the sum $X_1 + \ldots + X_n$ is normal as well, with mean $\mu_1 + \ldots + \mu_n$ and variance $\sigma_1^2 + \ldots + \sigma_n^2$.

  In general, we have $E[X + Y] = E[X] + E[Y]$ for arbitrary r.v.s X and Y. In contrast to this, the equality $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$ holds only if X and Y are uncorrelated (e.g. independent).