4 Pricing Exotic Options via Monte Carlo

The options considered up until now are termed *vanilla* options – they have standard, well defined properties and their prices are regularly quoted by exchanges or brokers. Here, we will consider some nonstandard *exotic* options whose prevalence has increased in recent years. In general, the payoff of these options at maturity depends not only on the stock price at that time but also on the path leading up to it. Consequently, explicit formulas (assuming a risk neutral G.B.M.) for these options can be difficult to find. We will therefore show how to use Monte Carlo simulation to compute their fair price. For simplicity (and to allow convenient Monte Carlo simulation) we will only consider options with fixed exercise times. The exotic options we will consider are:

- Power call options,
- Barrier options (calls and puts),
- Asian and lookback options (calls and puts).

4.1 Power Options

Recall that the standard European call option with strike price K has payoff at maturity T given by

$$\max(S_T - K, 0) = (S_T - K)^+ .$$

That is, the payoff is a linear function of stock price at maturity. More general call options exist, for example those with payoff given by

$$(S_T^\gamma - K)^+$$

are called *power* options and γ is called the power parameter (which in practice is a number).

Definition 4.1

A *power call option* gives the holder the right (but not obligation) to *buy* 1 share of stock for K and then immediately sell for market value raised to the power γ.

Pricing the power call option

It turns out that we can find an explicit formula for the risk-neutral G.B.M. valuation (fair price) of such options. This will be useful for comparing against estimation strategies later on.

Let $C_\gamma(S_0, T, K, \sigma^2, r)$ denote the price of the European call option with power γ. As usual, S_0 is the initial stock price, T is the maturity, K is the strike price, σ^2 is the volatility (under the assumption of G.B.M.) and r is the interest rate. Hence, $C_1(S_0, T, K, \sigma^2, r) = C(S_0, T, K, \sigma^2, r)$ denotes the Black-Scholes price of the vanilla call option, that is, with power parameter 1. In particular, note that the Black-Scholes price of one European call option is the *discounted expected payoff at maturity*

$$C_1(S_0, T, K, \sigma^2, r) \quad = \quad e^{-rT}\mathrm{E}\left[(S_T - K)^+\right]$$

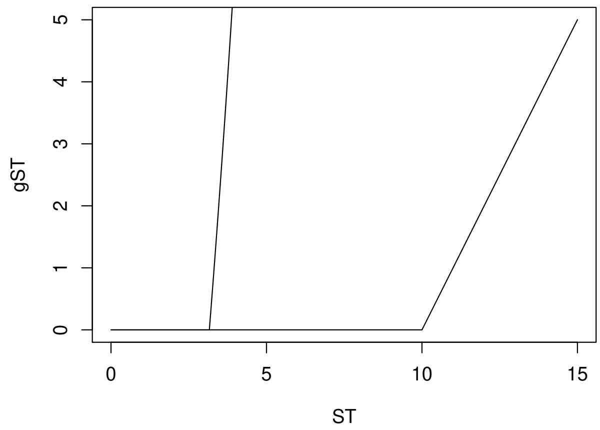

Figure 1: Power call option payoff with $K = 10$ and $\gamma = 2$ (quadratic after $\sqrt{10}$) and $\gamma = 1$ (linear after 10).

$$= e^{-rT}\mathrm{E}\left[(S_0\frac{S_T}{S_0} - K)^+\right]$$

where $S_T/S_0 \sim LN([r - \sigma^2/2]T, \sigma^2 T)$ under the assumption of risk-neutral G.B.M. We know that this expression can be evaluated to give the Black-Scholes formula of Section 2.

Now, the fair price of the power call option is

$$\begin{aligned} C_\gamma(S_0, T, K, \sigma^2, r) &= e^{-rT}\mathrm{E}\left[(S_T^\gamma - K)^+\right] \\ &= e^{-rT}\mathrm{E}\left[\left(S_0^\gamma\frac{S_T^\gamma}{S_0^\gamma} - K\right)^+\right]. \end{aligned}$$

Now, under the assumption that stock prices follow the risk neutral G.B.M. we know that S_T/S_0 has a log-normal distribution as above. Note that

$$\log\left(\frac{S_T^\gamma}{S_0^\gamma}\right) = \gamma\log\left(\frac{S_T}{S_0}\right) \sim N(\gamma[r - \sigma^2/2]T, \gamma^2\sigma^2 T).$$

We can *force* this distribution to have the same form as G.B.M. by defining new parameters

$$\sigma_\gamma^2 = \gamma^2\sigma^2, \qquad r_\gamma - \sigma_\gamma^2/2 = \gamma(r - \sigma^2/2).$$

Hence,

$$\begin{aligned} \gamma\log\left(\frac{S_T}{S_0}\right) &\sim N([r_\gamma - \sigma_\gamma^2/2]T, \sigma_\gamma^2 T) \\ \Rightarrow \frac{S_T^\gamma}{S_0^\gamma} &\sim LN([r_\gamma - \sigma_\gamma^2/2]T, \sigma_\gamma^2 T). \end{aligned}$$

Now, we have $C_\gamma(S_0, T, K, \sigma^2, r)$ as

$$\begin{aligned} C_\gamma(S_0,T,K,\sigma^2,r) &= e^{-rT}\mathrm{E}\left[\left(S_0^\gamma\frac{S_T^\gamma}{S_0^\gamma}-K\right)^+\right] \\ &= e^{-rT}e^{r_\gamma T}e^{-r_\gamma T}\mathrm{E}\left[\left(S_0^\gamma\frac{S_T^\gamma}{S_0^\gamma}-K\right)^+\right] \\ &= e^{-rT}e^{r_\gamma T}C_1(S_0^\gamma,T,K,\sigma_\gamma^2,r_\gamma) \end{aligned}$$

where the last line follows by using the *form* of the Black-Scholes price (as a discounted expected payoff with respect to a G.B.M., where in this case S_0 is replaced by S_0^γ, r by r_γ and σ by σ_γ). Hence we have found an explicit formula for the fair price of a power call option with parameter γ. Let's see how this works with an example.

Example 4.1

Find the fair (risk-neutral) price of a power call option with maturity $T = 2$ years, strike price £3000, initial stock price $S_0 =$ £50 and payoff, $\max(S_T^2 - K, 0)$ (a power parameter of 2). Suppose further that the risk-free interest rate is $r = 0.05$ and stock price follows a geometric Brownian motion with volatility parameter $\sigma^2 = 0.01$.

Solution

The fair price of the option is

$$C_\gamma(S_0,T,K,\sigma^2,r) = e^{-rT}e^{r_\gamma T}C_1(S_0^\gamma,T,K,\sigma_\gamma^2,r_\gamma)$$

where $S_0 = 50$, $T = 2$, $K = 3000$, $\sigma^2 = 0.01$, $r = 0.05$, $\gamma = 2$, $S_0^2 = 2500$, $\sigma_\gamma^2 = \gamma^2\sigma^2 = 0.04$ *and*

$$\begin{aligned} r_\gamma - \sigma_\gamma^2/2 &= 0.09 \\ \Rightarrow r_\gamma &= 0.09 + 0.02 = 0.11\,. \end{aligned}$$

We compute $C_1(S_0^\gamma, T, K, \sigma_\gamma^2, r_\gamma)$ *using the Black-Scholes formula of Section 2,*

$$C_1(S_0^\gamma,T,K,\sigma_\gamma^2,r_\gamma) = 2500\Phi(\omega) - 3000e^{-0.11\times 2}\Phi(\omega - 0.2\sqrt{2})$$

where

$$\omega = \frac{0.11\times 2 + 0.2^2\times 2/2 - \log(3000/2500)}{0.2\sqrt{2}} = 0.2746$$

and recall that log *is base e. Hence we obtain* $C_1(S_0^\gamma, T, K, \sigma_\gamma^2, r_\gamma) = 324.59$. *Therefore the fair price of the power call option is*

$$\begin{aligned} C_\gamma(S_0,T,K,\sigma^2,r) &= e^{(0.05+2\times 0.1^2/2)2}\times 324.59 \\ &= 365.97. \end{aligned}$$

We will revisit this example later on, when looking at Monte Carlo pricing.

4.2 Barrier Options

Barrier options are options whose payoff depends on whether the asset's price reaches a certain level before maturity. To define a *European barrier call option* with strike price K and maturity T, we specify a barrier ν – depending on the type of barrier option, the option either comes alive or is killed when the barrier is breached.

Definition 4.2

There are several types of barrier option:

- A *down-and-in* barrier option gives the holder the right to exercise the option at time T provided that the stock price goes below ν at some time before T i.e. the option becomes *alive* only if the security's price goes *below* ν before T.
- A *down-and-out* barrier option is *killed* if the security's price goes *below* ν before T. Note that in both the down-and out and down-and-in options, ν is a value less than the initial stock price S_0.
- An *up-and-in* barrier option becomes *alive* only if the security's price goes *above* ν before T.
- An *up-and-out* barrier option is *killed* if the security's price goes *above* ν before T. Note that in both the up-and out and up-and-in options, ν is a value greater than the initial stock price S_0.

Illustration

Figure 2 shows an example of an up-and-out barrier call that either expires worthless (barrier breached) or not (barrier not breached and above strike at maturity).

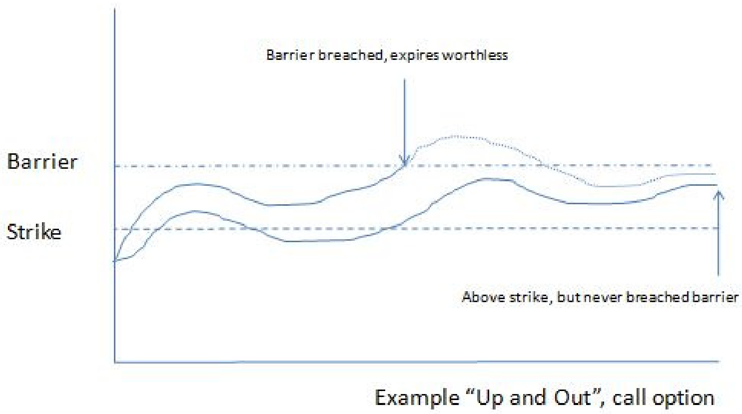

Figure 2: Up-and-out barrier call option. Two stock price path scenarios.

Comments

- Provided the option remains alive, the payoff at time T (for the European call) is $\max(S_T - K, 0)$. If the option is killed at any time, the payoff is 0.
- The same definitions exist for the European barrier put, the only difference being that if the option remains alive, the payoff at maturity is $\max(K - S_T, 0)$.
- If you own both a down-and-out and a down-and-in call option both the same strike price K and maturity T then this is equivalent to owning one vanilla call option (with parameters K and T). This is true since only one option can be in play at any time t (the down-and-in option if the barrier is breached and the down-and-out otherwise). Consequently, if we denote by C_{di} and C_{do} the respective risk neutral present values of owning the down-and-in and down-and-out call options, then

$$C_{di} + C_{do} = C$$

  where C is the Black-Scholes valuation of the vanilla European call option given in Section 2. A similar argument follows for the up-and-in and up-and-out options.
- We typically observe stock price on a daily basis. Therefore, let

$$S_i^d = S_{i/252}$$

  denote the price on day i at some arbitrary time. Hence, the down-and-in barrier call option has payoff

$$\begin{cases} (S_T - K)^+ & \text{if } S_i^d \le \nu \text{ for some } i = 1, \ldots, 252T \\ 0 & \text{if } S_i^d > \nu \text{ for all } i = 1, \ldots, 252T \end{cases}$$

  Similarly, the down-and-out call option has payoff

$$\begin{cases} 0 & \text{if } S_i^d \le \nu \text{ for some } i = 1, \ldots, 252T \\ (S_T - K)^+ & \text{if } S_i^d > \nu \text{ for all } i = 1, \ldots, 252T \end{cases}$$

4.3 Asian and Lookback Options

Asian options are options whose payoff depends on the average price of the asset during at least some part of the asset's lifetime. These averages are usually in terms of the daily closing prices and we therefore let

$$S_i^d = S_{i/252}$$

denote the price on day i as before. The most common Asian-type call option with strike price K and maturity T (in years) has the following definition.

Definition 4.3

The holder of the *Asian call* option has the right (but not obligation) to buy 1 share for K and sell for the average price realised over $(0, T]$.

Hence, assuming daily prices, the payoff is

$$\left(\sum_{i=1}^{252T} \frac{S_i^d}{252T} - K \right)^+ .$$

A common Asian put option with strike price K and maturity T has payoff

$$\left(K - \sum_{i=1}^{252T} \frac{S_i^d}{252T} \right)^+ .$$

Definition 4.4

The holder of the *lookback call* option has the right (but not obligation) to buy 1 share for

$$K = \min_{i=1,\ldots,252T} S_i^d$$

at time T.

Hence, the lookback call option with maturity T has strike price given by the minimum end-of-day price up to the maturity time. The payoff at time T is

$$S_T - \min_{i=1,\ldots,252T} S_i^d .$$

The *lookback put* has strike price given by the maximum end-of-day price up to the maturity time. Hence, the payoff is

$$\max_{i=1,\ldots,252T} S_i^d - S_T .$$

Note that because the payoffs of both the lookback and Asian type options depend on the price path followed, there are no known exact formulas for the risk-neutral valuations of these options. However, approximate valuations are possible by using Monte Carlo simulation methods.

4.4 Monte Carlo Integration

Suppose we have a random variable X with p.d.f. $f_X(x)$ and our goal is to estimate

$$\theta = \mathrm{E}(X) = \int_X x f_X(x) \, dx .$$

If we can generate values of $X_1, \ldots, X_N$ from $f_X(\cdot)$ then an unbiased estimator of the theoretical mean $\theta = \mathrm{E}(X)$ is given by the sample mean

$$\bar{X} = \frac{1}{N} \sum_{i=1}^{N} X_i .$$

Plainly,

$$\mathrm{E}(\bar{X}) = \frac{1}{N} \sum_{i=1}^{N} \mathrm{E}(X_i) = \theta .$$

Now suppose that $\mathrm{Var}(X) = v^2$ then

$$\mathrm{Var}(\bar{X}) = \frac{1}{N^2}\sum_{i=1}^{N}\mathrm{Var}(X_i) = v^2/N\,.$$

Hence we have shown that $\bar{X}$ is both unbiased and consistent. Of course this argument can be generalised. Suppose our goal is to evaluate $\mathrm{E}(g(X))$ for some function $g(\cdot)$. By definition,

$$\mathrm{E}(g(X)) = \int_X g(x) f_X(x)\,dx\,.$$

However, if we cannot perform the integration, how can we proceed? Again, if we can generate values of $X_1, \ldots, X_N$, then an unbiased estimator of $\mathrm{E}(g(X))$ is given by

$$\frac{1}{N}\sum_{i=1}^{N} g(X_i)$$

and this estimator has variance proportional to $1/N$. Therefore, better estimates (in the sense of a small variance) are obtained for large N. This approach is known as Monte Carlo simulation and can be applied to the pricing problem as follows.

4.5 Pricing via Simulation

Suppose we are interested in finding the fair (risk-neutral) price of an option with payoff $g(\cdot)$ at maturity T (in years) depending on daily stock price $S_1^d, \ldots, S_{252T}^d$. If the risk free interest rate is r, then this price is given by the discounted expected payoff at maturity. This is exactly the problem described above, in the sense that the expectation involves integration, and so we can apply Monte carlo simulation. Algorithmically, we perform the following

1. Simulate a random path $S_0^d, S_1^d, \ldots, S_{252T}^d$ in the risk-neutral world.
2. Calculate the payoff $g(\cdot)$ from the option at time T.
3. Repeat steps 1 and 2 to get say N sample values of the payoff.
4. Calculate the mean of these sample payoffs to get an estimate of the expected payoff.
5. Discount the expected payoff at the risk-free interest rate to get an estimate of the value of the option.

If we assume that stock prices follow a risk-neutral G.B.M. then we have already seen how to simulate this process in Section 2.

Example 4.1 revisited

Find the fair (risk-neutral) price of a power call option with maturity $T = 2$ years, strike price £3000, initial stock price $S_0 = £50$ and payoff, $\max(S_T^2 - K, 0)$ (a power parameter of 2). Suppose further that the risk-free interest rate is $r = 0.05$ and stock price follows a geometric Brownian motion with volatility parameter $\sigma^2 = 0.01$. Compare the analytic fair price to estimates obtained via Monte Carlo.

Solution

Recall that the exact fair price is 365.97. *We can also estimate the fair price of the option via Monte Carlo. A key step is the simulation of the stock price process. Recall from Section 2.3.1 that for an equally spaced partition of* $[0,T]$ *with time step* Δt

$$S_{t_i} = S_{t_{i-1}} \exp\left\{(r - \sigma^2/2)\Delta t + \sigma(W_{t_i} - W_{t_{i-1}})\right\}$$

where $W_{t_i} - W_{t_{i-1}} \sim N(0, \Delta t)$. *To simulate on a daily basis, we set* $\Delta t = 1/252$ *and, starting with* S_0^d *as a known value, simulate* $S_1^d, \ldots S_{252T}^d$ *via the recursion*

$$S_i^d = S_{i-1}^d \exp\left\{(r - \sigma^2/2)\frac{1}{252} + \sigma(W_i^d - W_{i-1}^d)\right\}, \quad i = 1, 2, \ldots, 252T$$

where $252T$ *is the maturity time in days. We then calculate a sample value of the expected payoff at time* T, *via*

$$X_1 = max\left((S_{252T}^d)^2 - 3000, 0\right) .$$

We repeat these steps a further $N-1$ *times to give* N *sample payoffs* $X_1, \ldots, X_N$. *We take the average and discount at the risk free interest rate to give an estimate of the fair price of the power option as*

$$e^{-rT}\frac{1}{N}\sum_{i=1}^{N} X_i .$$

The following R *function takes as arguments* S_0, K, T, γ, σ, r *and* N, *and returns the Monte Carlo estimate of the fair price of the power option.*

```
monte1=function(T=2,s0=50,r=0.05,sig=0.1,k=3000,N=1000)
{
 n=T*252
 s=vector("numeric",len=n+1)
 payoff=vector("numeric",len=N)
 for(j in 1:N){
  s[1]=s0
  for(i in 2:(n+1))
   {
    s[i]=s[i-1]*exp(rnorm(1,(r-0.5*sig*sig)/252,sig/sqrt(252)))
   }
  payoff[j]=max((s[n+1])^(2)-k,0)
 }
 exp(-r*T)*mean(payoff)
}

monte1()
```

A single execution of this function (with $N = 10000$*) gave an estimate of 372.10 which is in reasonable agreement with the actual price of 365.97. Note that the payoff is not dependent on the whole path, only the price of the stock at maturity. Since we know the distribution of* S_T *(log-normal), it is far more efficient to simulate* N *values of* S_T *and then set*

$$X_i = max\left((S_T)^2 - 3000, 0\right) \; i = 1, \ldots, N$$

before taking the discounted average as an estimate of the fair price. The required R *function is*

```
monte1=function(T=2,s0=50,r=0.05,sig=0.1,k=3000,N=1000)
{
 payoff=vector("numeric",len=N)
 for(j in 1:N)
 {
  s=s0*exp(rnorm(1,(r-0.5*sig*sig)*T,sig*sqrt(T)))
  payoff[j]=max(s^(2)-k,0)
 }
 exp(-r*T)*mean(payoff)
}

monte1()
```

Finally, note that every time we execute the function, we get a different estimate, since we're generating a realisation of the estimator, which is a random variable.

Example 4.2

Suppose that stock prices follow a G.B.M. with volatility parameter $\sigma^2 = 0.01$, the initial stock price is £50, the risk free interest rate is $r = 0.05$. Describe a detailed Monte Carlo algorithm to find the risk-neutral (fair) price of a down-and-in barrier call option with strike price $K = £51$, maturity $T = 1$ year and barrier $\nu = 49$.

Solution

Let us assume that the stock price is observed daily. *The fair price of the option is the discounted expected payoff at time T given by*

$$e^{-rT} E\left(I(S^d_{252} - K)^+\right)$$

where I is an indicator *function defined by*

$$I = \begin{cases} 1 & \text{if } S^d_i \le \nu \text{ for some } i = 1, \ldots, 252 \\ 0 & \text{if } S^d_i > \nu \text{ for all } i = 1, \ldots, 252 \end{cases}$$

That is, the option becomes alive (and remains alive) if the end-of-day stock price falls below ν at any time before maturity. We estimate the fair price C_{di} of the option by implementing the following sequence of steps:

1. *Simulate a random path $S^d_0, S^d_1, \ldots, S^d_{252T}$ in the risk-neutral world, via the recursion*

   $$S^d_i = S^d_{i-1} \exp\left\{(r - \sigma^2/2)\frac{1}{252} + \sigma(W^d_i - W^d_{i-1})\right\}, \quad i = 1, 2, \ldots, 252$$

   where $W^d_i - W^d_{i-1} \sim N(0, 1/252)$.

2. *Calculate a sample payoff at time T with*

   $$X_1 = I \times max(0, S^d_{252} - 51).$$

3. *Repeat steps 1 and 2 to get say N sample values of the payoff,*

$$X_1, X_2, \ldots, X_N$$

4. *Calculate the mean of these sample payoffs to get an estimate of the expected payoff.*

5. *Discount the expected payoff at the risk-free interest rate to get an estimate of the value of the option, that is, calculate*

$$e^{-rT}\bar{X}.$$

In practice, this is achieved in R *with (for example) the following function.*

```
monte2=function(T=1,s0=50,r=0.05,sig=0.1,k=51,nu=49,N=1000)
{
 n=T*252
 s=vector("numeric",len=n+1)
 payoff=vector("numeric",len=N)
 for(j in 1:N){
  s[1]=s0
  I=0
  for(i in 2:(n+1))
   {
    s[i]=s[i-1]*exp(rnorm(1,(r-0.5*sig*sig)/252,sig/sqrt(252)))
    if(s[i]<nu)
    {
     I=1
    }
   }
  payoff[j]=max(I*(s[n+1]-k),0)
 }
 exp(-r*T)*mean(payoff)
}

monte2()
```

A single call of this function with $N = 10000$ gave an estimate of C_{di} as 1.24. We can use this value to estimate the fair price C_{do} of the down-and-out barrier call option (with the same parameters) by using the relation in the comments on page 54. We calculate the fair price of the vanilla call option (with the same parameters) via the Black-Scholes formula. Performing the desired calculation gives $C = 2.80$. Hence an estimate of C_{do} is 1.56.

Comments

- Note that between any two time instants t_i and t_{i+1} at which we observe the stock, S_t could fall below ν but then increase sufficiently to become greater than ν before t_{i+1}. This breaking of the barrier would go undetected as we only have the sample path at discrete time intervals. However, provided we use a sufficiently fine discretisation, the probability of this occurring is very small. Hence, the error introduced from discrete sampling of the path is small.

4.6 Variance Reduction Techniques

Consider the task of choosing the number of simulated payoffs, N, in the Monte-Carlo estimate of the fair price of a particular option with payoff function $g(\cdot)$. Denote the fair price by ϕ and its Monte Carlo estimator by $\hat{\phi}$. Hence,

$$\phi = e^{-rT}\mathrm{E}\{g(\cdot)\}$$

and the Monte Carlo estimator is

$$\hat{\phi} = \frac{e^{-rT}}{N}\sum_{i=1}^{N} X_i$$

where X_i denotes a sample payoff and the collection of X_i are iid. Note that $\mathrm{E}(\hat{\phi}) = \phi$ and let $\mathrm{Var}(e^{-rT}X_i) = v^2$ so that $\mathrm{Var}(\hat{\phi}) = v^2/N$.

For large N, the central limit theorem applies and

$$\hat{\phi} \sim \mathrm{N}(\phi\,,\, v^2/N) \qquad \text{approximately.}$$

Hence, a 95% confidence interval is given by

$$\hat{\phi} - 1.96\frac{v}{\sqrt{N}} < \phi < \hat{\phi} + 1.96\frac{v}{\sqrt{N}}$$

Hence our uncertainty about the value of the fair price is inversely proportional to $\sqrt{N}$. Therefore, to double the accuracy of a simulation, we must quadruple N; to increase the accuracy by 10, the number of trials must be increased by a factor of 100.

To estimate the fair price by Monte-Carlo, we therefore typically need a very large value of N to ensure reasonable accuracy. This can be very costly in terms of computation time. We therefore examine a very simple technique that reduces the variance of the estimator for given N.

4.6.1 Using Antithetic Variables

Recall that a realisation of the daily stock price process can be generated via the recursion

$$S_i^d = S_{i-1}^d \exp\left\{\frac{(r-\sigma^2/2)}{252} + \sigma(W_i^d - W_{i-1}^d)\right\} \qquad i = 1,\ldots,252T$$

where $W_i^d - W_{i-1}^d \sim \mathrm{N}(0, 1/252)$. It will be helpful for us to re-write this equivalently as

$$S_i^d = S_{i-1}^d \exp\{Y_i\} \qquad \text{where} \qquad Y_i \sim \mathrm{N}\left(\frac{(r-\sigma^2/2)}{252}\,,\,\frac{\sigma^2}{252}\right).$$

Hence, in step 1 of the Monte Carlo algorithm we generate $Y_1,\ldots,Y_{252T}$ and use these values to compute $S_1^d,\ldots,S_{252T}^d$. In step 2, we calculate a realisation of the payoff $X_1 = g(S_1^d,\ldots,S_{252T}^d)$.

The anithetic technique re-uses / recycles the Y_i in a clever way to calculate a second stock price realisation, and in turn, a second payoff X_2 that is *negatively correlated* with X_1. To this end, the anithetic technique sets

$$Y_i^* = \frac{2(r-\sigma^2/2)}{252} - Y_i \qquad \text{for} \qquad i = 1,\ldots,252T$$

and uses the Y_i^* to generate a new realisation of the price process, $S_1^{d*}, \ldots, S_{252T}^{d*}$, before finally computing $X_2 = g(S_1^{d*}, \ldots, S_{252T}^{d*})$. The process then repeats to calculate pairs of negatively correlated payoffs X_3 and X_4, etc.

Is this a valid Monte Carlo strategy? Yes. Note that the Y_i^* have the same distribution as the Y_i but are negatively correlated with the Y_i. To see this, note that Y_i^* is just a linear transformation of Y_i and

$$\begin{aligned} \mathrm{E}(Y_i^*) &= \frac{2(r - \sigma^2/2)}{252} - \mathrm{E}(Y_i) = \mathrm{E}(Y_i), \\ \mathrm{Var}(Y_i^*) &= (-1)^2 \mathrm{Var}(Y_i) = \mathrm{Var}(Y_i). \end{aligned}$$

Finally,

$$\begin{aligned} \mathrm{Cov}(Y_i^*, Y_i) &= \mathrm{Cov}\left(\frac{2(r - \sigma^2/2)}{252} - Y_i, Y_i\right) \\ &= -\mathrm{Var}(Y_i) \\ &< 0. \end{aligned}$$

This negative covariance induces a negative covariance between X_1 and X_2, X_3 and X_4 etc. Consequently, for the antithetic scheme,

$$\mathrm{Var}(\bar{X}) = \frac{1}{N^2} \sum_{i=1}^{N} \mathrm{Var}(X_i) + \frac{2}{N^2} \sum_{i<j} \mathrm{Cov}(X_i, X_j)$$

and we note that the first term is the variance of the estimator used in the *standard* Monte Carlo scheme and the second term is negative. Hence, comapared to the standard scheme, the antithetic scheme gives an estimator with a smaller variance.